ON THE FURSTENBERG MEASURE AND DENSITY OF STATES FOR THE ANDERSON-BERNOULLI MODEL AT SMALL DISORDER

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Abstract. We establish new results on the dimension of the Furstenberg measure and the regularity of the integrated density of states for the Anderson-Bernoulli model at small disorder.

§0. Summary

Let $H = \Delta + \lambda V$ where Δ is the lattice Laplacian on $\mathbb{Z}$ and $V = (V_n)_{n\in\mathbb{Z}}$ are independent random variables in $\{1, -1\}$. We assume $|\lambda|$ small and restrict the energy E outside a fixed neighborhood of $\{0, 2, -2\}$. It is shown that the Furstenberg measure ν_E of the corresponding $SL_2(\mathbb{R})$-cocycle $\begin{pmatrix} E-\lambda V_n & -1 \\ 1 & 0 \end{pmatrix}$ has dimension at least $\gamma(\lambda)$, where $\gamma(\lambda) \overset{\lambda\to 0}{\longrightarrow} 1$. As a consequence, we derive that the integrated density of states (IDS) $\mathcal{N}(E)$ is Hölder-regular with exponent at least $s(\lambda) \overset{\lambda\to 0}{\longrightarrow} 1$.

The spectral theory of the Anderson-Bernoulli (A-B) model has been studied by various authors. It was shown by Halperin (see [S-T]) that for fixed $\lambda > 0$, $\mathcal{N}(E)$ is not Hölder continuous of any order α larger than

$$\alpha_0 = \frac{2\log 2}{\text{Arc}\ \cosh\ (1+\lambda)}. \tag{0.1}$$

Hölder regularity for some $\alpha > 0$ has been established in several papers. In [Ca-K-M], le Page's method is used. Different approaches (including one using the super-symmetric formalism) appear in the important paper [S-V-W] that relies on harmonic analysis principles around the uncertainty principle. In [B1], the author proved Hölder regularity of the IDS using the Figotin-Pastur expansion of the Lyapounov exponent and martingale theory. We note that in both [S-V-W] and [B1], the Hölder exponent α remains uniform for $\lambda \to 0$ (in fact, [B1] gives an explicit exponent $\alpha(\lambda) > \frac{1}{5} + \varepsilon$ for $\lambda \to 0$).

Thus the result in this Note just falls short of establishing the conjectured Lipschitz regularity of IDS of the A-B model for small λ. Related is the question whether the Furstenberg measure on projective space is absolutely continuous when λ is small (or even better). As pointed out at the end of the paper, a natural approach to these problems is through certain spectral gap properties that do not depend on hyperbolicity. There have been recent advances (cf. [BG1], [BG2], [B2]), that are based on methods from arithmetic combinatorics. But presently, this theory seems to restrictive for an application to A-B-cocycles. It does apply however for Schrödinger operators with single site distribution given by a measure of positive dimension.

§1 Probabilistic inequalities on the Boolean cube

The following statement is a consequence of Sperner's combinatorial Lemma.*

Lemma 1. *Let $f = f(\varepsilon_1, \dots, \varepsilon_n)$ be a real valued function on $\{1, -1\}^n$ and denote*

$$I_j = f|_{\varepsilon_j=1} - f|_{\varepsilon_j=-1} \tag{1.1}$$

the j-influence, which is a function of $\varepsilon_{j'}, j' \neq j$.

Assume that for all $j = 1, \dots, n$

$$I_j \geq 0 \tag{1.2}$$

(i.e. f is monotone increasing) and moreover

$$I_j \geq \kappa > 0 \ \text{ on } \ \Omega_j \cap \Omega_j' \tag{1.3}$$

where Ω_j (resp. Ω_j') are subsets of $\{1, -1\}^n$ depending only on the variables $\varepsilon_1, \dots, \varepsilon_{j-1}$ (resp. $\varepsilon_{j+1}, \dots, \varepsilon_n$). Then, for any $t \in \mathbb{R}$, we have

$$\text{mes}\,[|f - t| < \kappa] \leq \frac{1}{\sqrt{n}} + \sum_j (2 - \text{mes}\,\Omega_j - \text{mes}\,\Omega_j'). \tag{1.4}$$

Proof.

Denote

$$\tilde{\Omega} = \bigcap_{1 \leq j \leq n} (\Omega_j \cap \Omega_j') \tag{1.5}$$

*It was also used in [B1] and [B-K] in the context of the Anderson-Bernoulli model.

for which

$$1 - \operatorname{mes} \tilde{\Omega} \leq \sum_j (2 - \operatorname{mes} \Omega_j - \operatorname{mes} \Omega'_j). \tag{1.6}$$

We claim that the set $[|f - t| < \kappa] \cap \tilde{\Omega}$ does not contain a pair of distinct comparable elements $\varepsilon = (\varepsilon_j)_{1 \leq j \leq n}$ and $\varepsilon' = (\varepsilon'_j)_{1 \leq j \leq n}$. Assume otherwise and $\varepsilon < \varepsilon'$, i.e. $\varepsilon_j \leq \varepsilon'_j$ for each j. Then

$$\begin{aligned} f(\varepsilon') - f(\varepsilon) &= \sum_{1 \leq j \leq n} \big(f(\varepsilon_1, \ldots, \varepsilon_{j-1}, \varepsilon'_j, \ldots, \varepsilon'_n) - f(\varepsilon_1, \ldots, \varepsilon_j, \varepsilon'_{j+1}, \ldots, \varepsilon'_n) \big) \\ &= \sum_{\substack{1 \leq j \leq n \\ \varepsilon_j \neq \varepsilon'_j}} I_j(\varepsilon_1, \ldots, \varepsilon_j, \varepsilon'_{j+1}, \ldots, \varepsilon'_n). \end{aligned} \tag{1.7}$$

Since $\varepsilon \in \Omega_j, \varepsilon' \in \Omega'_j$, it follows from our assumption on $\Omega_j, \Omega_{j'}$ that

$$(\varepsilon_1, \ldots, \varepsilon_j, \varepsilon'_{j+1}, \ldots, \varepsilon'_n) \in \Omega_j \cap \Omega'_j$$

and hence $I_j(\varepsilon_1, \ldots, \varepsilon_j, \varepsilon'_{j+1}, \ldots \varepsilon'_n) \geq \kappa$ by (1.3). In particular, since $\varepsilon \neq \varepsilon'$,

$$(1.7) \geq \#\{1 \leq j \leq n; \varepsilon_j \neq \varepsilon'_j\}\kappa \geq \kappa$$

which is however impossible if $|f(\varepsilon) - t| \leq \kappa$ and $|f(\varepsilon') - t| \leq \kappa$. This establishes the claim.

Therefore, by Sperner's lemma on the maximal size of subsets of $\{1, -1\}^n$ not containing any pair of distinct comparable elements, we get

$$\operatorname{mes} (\tilde{\Omega} \cap [|f - t| < \kappa]) \lesssim \frac{1}{\sqrt{n}} \tag{1.8}$$

and (1.4) follows from (1.6), (1.8). This proves Lemma 1.

We will use the following corollary of Lemma 1.

Lemma 2. *Let f and I_j be as in Lemma 1 and assume each $I_j \geq 0$.*

Assume further $\kappa, \delta > 0$ and for each $1 \leq j < n$

$$f|_{\varepsilon_j = 1, \varepsilon_{j+1} = 1} - f|_{\varepsilon_j = -1, \varepsilon_{j+1} = -1} \geq \kappa \text{ for } \varepsilon \in \Omega_j \tag{1.9}$$

where $\Omega_j \subset \{1, -1\}^n$ is a set only depending on the variables $\varepsilon_{j+2}, \ldots, \varepsilon_n$ and such that

$$\operatorname{mes} \Omega_j > 1 - \delta. \tag{1.10}$$

Then, for all $t \in \mathbb{R}$

$$\operatorname{mes}[|f-t|<\kappa] \lesssim \frac{1}{\sqrt{n}} + n\delta. \tag{1.11}$$

Proof. Assume $n = 2m$ even and write $\omega = (\varepsilon_1, \varepsilon_1', \dots, \varepsilon_m, \varepsilon_m')$ for the $\{1,-1\}^n$-variable. With this notation, let Ω_j refer to the set Ω_{2j-1}.

Partition

$$\{1,-1\}^{2m} = \bigcup_{S\subset\{1,\dots,m\}} V_S$$

with

$$V_S = \{\omega; \varepsilon_j = \varepsilon_j' \text{ if } \ j\in S \text{ and } \ \varepsilon_j \neq \varepsilon_j' \text{ if } \ j \notin S\}. \tag{1.12}$$

Thus

$$\operatorname{mes}[|f-t|<\kappa] = \sum_{S\subset\{1,\dots,m\}} \operatorname{mes}[V_S \cap |f-t|<\kappa]. \tag{1.13}$$

Fix $S \subset \{1,\dots,m\}$.

We consider f on V_S as a function of $(\varepsilon_j)_{j\in S}$ with the other variables $(\varepsilon_j, \varepsilon_j')_{j\in S}$ fixed. Denoting $g = g(\varepsilon_j; j\in S)\}$ this function on $\{1,-1\}^{|S|}$, we have by our assumption (1.9), for $j \in S$

$$\begin{aligned}
I_j(g)(\varepsilon_j, j\in S) &= f(\varepsilon_1, \varepsilon_1', \dots, \varepsilon_{j-1}, \varepsilon_{j-1}', 1, 1, \varepsilon_{j+1}, \varepsilon_{j+1}', \dots, \varepsilon_n, \varepsilon_n') \\
&\quad - f(\varepsilon_1, \varepsilon_1', \dots, \varepsilon_{j-1}, \varepsilon_{j-1}', -1, -1, \varepsilon_{j+1}, \dots, \varepsilon_n') \\
&\geq \kappa
\end{aligned}$$

provided

$$\begin{aligned}
(\varepsilon_k)_{k\in S} \in \Omega_j' &= \{(\varepsilon_k)_{k\in S}; \big((\varepsilon_k, \varepsilon_k)_{k\in S}, (\varepsilon_k, \varepsilon_k')_{k\notin S}\big) \in \Omega_j\} \\
&= (\Omega_j \cap V_S)\ (\varepsilon_k, \varepsilon_k'; k\notin S) \subset \{1,-1\}^{|S|}
\end{aligned}$$

hence only depending on $(\varepsilon_k)_{k\in S, k>j}$, (recall that we fixed the variables outside S).

Applying Lemma 1 to the function g (with $\Omega_j = \{1,-1\}^{|S|}$ for all $j\in S$), we obtain

$$\begin{aligned}
\#[\omega\in V_S; |f(\omega)-t|<\kappa] &\leq \frac{\#V_S}{|S|^{1/2}} + \sum_{j\in S} \sum_{\varepsilon_k\neq\varepsilon_k', k\notin S} \big(2^{|S|} - \#(\Omega_j\cap V_S)(\varepsilon_k, \varepsilon_k'; k\notin S)\big) \\
&= \frac{\#V_S}{|S|^{\frac{1}{2}}} + \sum_{j\in S} \#(V_S\backslash\Omega_j).
\end{aligned} \tag{1.14}$$

Summing over $S \subset \{1, \dots, m\}$ gives

$$(1.13) \leq 2^{-m} \sum_{S \subset \{1,\dots,m\}} \frac{1}{|S|^{\frac{1}{2}}} + \sum_{j=1}^{n} \text{mes}\,(\Omega \backslash \Omega_j)$$

$$\lesssim \left(\frac{m}{2}\right)^{-\frac{1}{2}} + n\delta \tag{1.15}$$

and hence (1.11).

§2. Application to the Anderson-Bernoulli model

Consider the projective action of $SL_2(\mathbb{R})$ on $P_1(\mathbb{R}) \simeq \mathbb{T} = \mathbb{R}/\mathbb{Z}$, defined for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ by

$$e^{i\tau_g(\theta)} = \frac{(a\cos\theta + b\sin\theta) + i(c\cos\theta + d\sin\theta)}{[(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2]^{1/2}}. \tag{2.1}$$

Hence

$$(\tau_g)'(\theta) = \frac{\sin^2\tau_g(\theta)}{(c\cos\theta + d\sin\theta)^2} = \frac{1}{[(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2]^{1/2}} \tag{2.2}$$

and

$$\|g\|^2 \geq (\tau_g)' \geq \frac{1}{\|g\|^2}. \tag{2.3}$$

Consider the Anderson-Bernoulli model (A-B model)

$$H_\lambda(\varepsilon) = \lambda\varepsilon_n\delta_{nn'} + \Delta \tag{2.4}$$

with $\varepsilon = (\varepsilon_n)_{n\in\mathbb{Z}} \in \{1, -1\}^{\mathbb{Z}}$ at small disorder $\lambda > 0$ (Δ stands for the usual lattice Laplacian).

The corresponding transfer operators $M_N(E) \in SL_2(\mathbb{R})$ are given by

$$M_N = M_N(E;\varepsilon) = \begin{pmatrix} E - \lambda\varepsilon_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \lambda\varepsilon_{N-1} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - \lambda\varepsilon_1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \prod_{N}^{1} g_{{}_E}(\varepsilon_j). \tag{2.5}$$

Considering ε_j $(1 \leq j \leq N)$ as a continuous variable on $[-1,1]$, and ∂_j the corresponding partial derivative, we have for the projective action

$$\tau_{M_N} = \tau_{g_E(\varepsilon_N)\cdots g_E(\varepsilon_{j+1})^o} \, \tau_{g_E(\varepsilon_j)^o} \tau_{g_E(\varepsilon_{j-1})\cdots g_E(\varepsilon_1)}$$

and

$$(\partial_j \tau_{M_N})(\theta) = \tau'_{g_E(\varepsilon_N)\cdots g_E(\varepsilon_{j+1})}(\tau_{g_E(\varepsilon_j)\cdots g_E(\varepsilon_1)}(\theta)).(\partial_j \tau_{g_E})(\tau_{g_E(\varepsilon_{j-1})\cdots g_E(\varepsilon_1)}). \tag{2.6}$$

Since

$$\operatorname{cotg} \tau_{g_E(\varepsilon)}(\theta) = (E - \lambda\varepsilon) - \frac{\sin\theta}{\cos\theta} \tag{2.7}$$

we have

$$(\partial_\varepsilon \tau_{g_E})(\theta) = \lambda. \sin^2 \tau_{g_E(\varepsilon)}(\theta) = \lambda \frac{\cos^2\theta}{\cos^2\theta + ((E-\lambda\varepsilon)\cos\theta - \sin\theta)^2} \sim \lambda \cos^2\theta. \tag{2.8}$$

From (2.3), (2.6), (2.8)

$$\begin{aligned}(\partial_j \tau_{M_N})(\theta) &\gtrsim \frac{\lambda}{\|g_E(\varepsilon_N)\cdots g_E(\varepsilon_{j+1})\|^2} \cos^2 \tau_{g_E(\varepsilon_{j-1})\cdots g_E(\varepsilon_1)}(\theta) \\ &= \frac{\lambda}{\|M_{N-j}(E;\varepsilon_{j+1},\dots,\varepsilon_N)\|^2} \cos^2 \tau_{M_{j-1}(E,\varepsilon)}(\theta). \end{aligned} \tag{2.9}$$

In order to deal with the issue of $\cos \tau_{M_{j-1}(E,\varepsilon)}(\theta)$ being small, note that by (2.7), for all θ, ε

$$|\cos\theta| + |\cos \tau_{g_E(\varepsilon)}(\theta)| > c. \tag{2.10}$$

Hence (2.9) implies

$$(\partial_j \tau_{M_N})(\theta) + (\partial_{j+1}\tau_{M_N})(\theta) \gtrsim \frac{\lambda}{\|M_{N-j}(E;\varepsilon_{j+1,\dots,\varepsilon_N})\|^2} \tag{2.11}$$

(for all θ).

In order to fulfill condition (1.9), we need an upperbound on $\|M_n(E;\varepsilon_1,\dots,\varepsilon_n)\|$. This function can be analyzed using the Figotin-Pastur expansion.

Denote

$$E = 2\cos\kappa \qquad (0 \leq \kappa \leq \pi) \tag{2.12}$$

$$V_n = -\frac{\varepsilon_n}{\sin\kappa} \tag{2.13}$$

where we assume $\delta_0 < |E| < 2 - \delta_0$ and hence κ stays away from $0, \frac{\pi}{2}, \pi$ (here δ_0 will be a fixed constant independent of λ).

The Figotin-Pastur formula gives

$$\frac{1}{N} \log \|M_N(E,\varepsilon)\| = \frac{1}{2N} \sum_1^N \log\left(1 + \lambda V_n \sin 2(\varphi_n + \kappa) + \lambda^2 V_n^2 \sin^2(\varphi_n + \kappa)\right) \tag{2.14}$$

with

$$\zeta_n = e^{2i\varphi_n} \tag{2.15}$$

recursively given by

$$\zeta_{n+1} = \mu\zeta_n + i\frac{\lambda}{2} V_n \frac{(\mu\zeta_n - 1)^2}{1 - \frac{i\lambda}{2} V_n(\mu\zeta_n - 1)} \tag{2.16}$$

and

$$\mu = e^{2i\kappa}. \tag{2.17}$$

Note that by (2.13), (2.16), ζ_n only depends on $\varepsilon_{n'}$ for $n' \leq n-1$.

Expanding (2.14), we obtain

$$(2.14) = \frac{\lambda^2}{8N} \sum_1^N V_n^2 \tag{2.18}$$

$$+ \frac{\lambda}{2N} \sum_1^N V_n \sin(\varphi_n + \kappa) \tag{2.19}$$

$$- \frac{\lambda^2}{4N} \sum_1^N V_n^2 \cos 2(\varphi_n + \kappa) \tag{2.20}$$

$$+ \frac{\lambda^2}{8N} \sum_1^N V_n^2 \cos 4(\varphi_n + \kappa) \tag{2.21}$$

$$+ 0(\lambda^3)$$

and

$$(2.18) = \frac{\lambda^2}{8\sin^2\kappa} = \frac{\lambda^2}{2(4-E^2)}. \tag{2.22}$$

By (2.16), (2.17)

$$|1-\mu| \left|\sum_1^N \zeta_n\right| < 1 + 0(\lambda N)$$

$$\left|\sum_1^N \xi_n\right| < \frac{0(\lambda N)}{\sin^2\kappa} \tag{2.23}$$

and similarly

$$\Big|\sum_1^N \zeta_n^2\Big| < \frac{0(\lambda N)}{\sin^2 2\kappa} < 0(\lambda N). \tag{2.24}$$

Writing $\cos 2(\varphi_n+\kappa) = \frac{1}{2}(\mu\zeta_n+\bar\mu\bar\zeta_n), \cos 4(\varphi_n+\kappa) = \frac{1}{2}(\mu^2\zeta_n^2+\bar\mu^2\bar\zeta_n^2)$, (2.23), (2.24) imply

$$(2.20), (2.21) = 0(\lambda^3). \tag{2.25}$$

Write

$$\begin{aligned}(2.19) &= \frac{-\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n \sin(\varphi_n+\kappa)\\ &= -\frac{\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n d_n(\varepsilon_{n'}; n'<n)\end{aligned} \tag{2.26}$$

which is a martingale difference sequence, with

$$\begin{aligned}\sum_1^N |d_n|^2 = \sum_1^N \sin^2 2(\varphi_n+\kappa) &< \frac{N}{2} + \frac{1}{2}\Big|\sum_1^N \mu^2\zeta_n^2 + \bar\mu^2\bar\zeta_n^2\Big|\\ &< \Big(\frac{1}{2}+0(\lambda)\Big)N.\end{aligned} \tag{2.27}$$

In conclusion

$$\frac{1}{N}\log\|M_N(E;\varepsilon)\| = \frac{\lambda^2}{8\sin^2\kappa} - \frac{\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n d_n + 0(\lambda^3) \tag{2.28}$$

and the Lyapounov exponent

$$L(E) = \frac{\lambda^2}{8\sin^2\kappa} + 0(\lambda^3). \tag{2.29}$$

From martingale theory and (2.28), we get for $a>0$ the large deviation inequality

$$\begin{aligned}\text{mes}\Big[\varepsilon\ \ \Big|\frac{1}{N}\log\|M_N(E;\varepsilon)\| - L(E)\Big| > aL(E)\Big] &< e^{-\big(\frac{a^2\lambda^2}{16\sin^2\kappa}+0(\lambda^3)\big)N}\\ &< e^{-\big(\frac{a^2}{2}L(E)+0(\lambda^3)\big)N}.\end{aligned} \tag{2.30}$$

In particular, taking $a > 2$ (and λ small)

$$\text{mes}\,[\varepsilon |\log \|M_N(E;\varepsilon)\| > aN\lambda^2] < e^{-ca^2\lambda^2 N}. \tag{2.31}$$

Returning to (2.11), take

$$N \sim \lambda^{-2}. \tag{2.32}$$

For $1 \leq j \leq N$, it follows from (2.31) that

$$\begin{aligned} &\text{mes}\,[(\varepsilon_{j+1}, \dots, \varepsilon_N); \|M_{N-j}(E; \varepsilon_{j+1}, \dots, \varepsilon_N)\| > e^{C_1(\log N)^{\frac{1}{2}}}] \leq \\ &\exp\{[-cC_1^2\lambda^2 \frac{\log N}{(\lambda^2(N-j))^2} + 0(\lambda^3)](N-j)\} < e[-cC_1^2 \log N + 0(\lambda)] < N^{-C_1}. \end{aligned} \tag{2.33}$$

Recalling (2.11), we see that Lemma 2 may be applied to the function

$$f = \tau_{M_N(E;\varepsilon)}(\theta) \text{ of } \varepsilon \in \{1, -1\}^N \text{ with } \kappa \sim \lambda e^{-2C_1(\log N)^{1/2}} \text{ and } \delta < N^{-C_1} < N^{-10},$$

for a suitable choice of the constant C_1.

Hence, we proved

Lemma 3. *For λ small and $N \sim \lambda^{-2}$, we have for fixed $\delta_0 < |E| < 2 - \delta_0$ and $\theta \in \mathbb{T}$ arbitrary, the distributional inequality*

$$\text{mes}\,[\varepsilon; |\tau_{M_N(\varepsilon)}(\theta) - t| < \lambda e^{-C|\log \lambda|^{\frac{1}{2}}}] \leq C\lambda \tag{2.34}$$

for all t (where C is some constant).

§3. Dimension of the Furstenberg measure

Fixing E as above, denote $\nu_E = \nu$ the Furstenberg measure on $\mathbb{T}$ for the random walk associated with the probability measure on $SL_2(\mathbb{R})$

$$\mu = \frac{1}{2}\,\delta_{\begin{pmatrix} E-\lambda & -1 \\ 1 & 0 \end{pmatrix}} + \frac{1}{2}\,\delta_{\begin{pmatrix} E+\lambda & -1 \\ 1 & 0 \end{pmatrix}}. \tag{3.1}$$

Thus for all N

$$\int_{SL_2(\mathbb{R})} \varphi(g)\mu^{(N)}(dg) = \int_{\{1,-1\}^N} \varphi\big(M_N(E,\varepsilon)\big)d\varepsilon. \tag{3.2}$$

The measure ν is μ-stationary i.e.

$$\nu = \int (\tau_g)_*[\nu]\mu(dg) \tag{3.3}$$

and

$$\langle \nu, f\rangle = \lim_{N\to\infty} \int f\big(\tau_{M_N(\varepsilon)}(\theta)\big)d\varepsilon \tag{3.4}$$

for all $f \in C(\mathbb{T})$ and $\theta \in \mathbb{T}$.

Our goal is to show that for small λ, the dimension of ν_E is close to 1.

The main inequality is the following

Lemma 4. *Let $h \in SL_2(\mathbb{R})$ be arbitrary such that*

$$\|h\| \sim \lambda^{-\frac{1}{10}}.$$

Let $N \sim \lambda^{-1}$ and $I \subset \mathbb{T}$ an arbitrary interval of size $|I| < \lambda$. Then

$$\int (\tau_{M_N(\varepsilon)h})_*[\nu](I)d\varepsilon \leq$$
$$e^{C|\log\lambda|^{\frac{1}{2}}}\Big\{ \max_{|J|<\lambda^{1/10}|I|} \nu(J) + \lambda^{\frac{1}{30}} \max_{|J|\leq |I|} \nu(J) + \max_{\lambda^{-\frac{1}{10}}<D<\lambda^{-\frac{1}{5}}} \frac{1}{D} \max_{|J|<D.|I|} \nu(J)\Big\} \tag{3.5}$$

with J denoting an interval.

Proof. Write

$$\int \big(M_N(\varepsilon)h\big)_*[\nu](I)d\varepsilon = \sum_{0\leq k\lesssim N} \int_{[\|M_N(\varepsilon)\|\sim 2^k]} \nu\big(\tau_{h^{-1}}\tau_{M(\varepsilon)^{-1}}(I)\big)d\varepsilon. \tag{3.6}$$

From (2.31)

$$\text{mes}\,[\|M_N(\varepsilon)\| \sim 2^k] < e^{-ck^2} \tag{3.7}$$

and, if $\|M_N(\varepsilon)\| \sim 2^k, \tau_{h^{-1}}\tau_{M_N(\varepsilon)^{-1}}(I)$ is contained in an interval $J \in \mathbb{T}$ of size at most $\|h\|^2 4^k |I|$. Thus the k^{th} summands in (3.6) is certainly bounded by

$$e^{-ck^2} \max_{|J|<4^k\|h\|^2|I|} \nu(J). \tag{3.8}$$

Next, restrict $k \lesssim (\log N)^{\frac{1}{2}}$ and ε to $[\|M_N(\varepsilon)\| \sim 2^k]$.

Let $R_1, \dots, R_M$ be a partition of $\mathbb{T}$ in intervals of size $\frac{1}{M} \sim \lambda$. Estimate

$$\int_{[\|M_N(\varepsilon)\|\sim 2^k]} \nu\big(\tau_{h^{-1}}\tau_{M_N(\varepsilon)^{-1}}(I)\big) d\varepsilon \leq$$

$$\sum_{m=1}^{M} \int_{[\|M_N(\varepsilon)\|\sim 2^k]} \nu\big(\tau_{h^{-1}}(\tau_{M_N(\varepsilon)^{-1}}(I)\cap R_M)\big) d\varepsilon$$

$$\leq \sum_{m=1}^{M} \text{mes}\,[\varepsilon; \|M_N(\varepsilon)\| \sim 2^k \text{ and } \tau_{M_N(\varepsilon)}(R_m)\cap I \neq \phi] \begin{bmatrix} \max \nu(J) \\ |J| \leq 4^k D_m |I| \end{bmatrix} \tag{3.9}$$

denoting

$$D_m = \max_{\theta\in R_m} |\tau'_{h^{-1}}(\theta)|. \tag{3.10}$$

Fixing some $\theta_m \in R_m$ and $\psi \in I$, $\tau_{M_N(\varepsilon)}(R_m)$ is contained in an $4^k\frac{1}{M}$-neighborhood of $\tau_{M_N(\varepsilon)}(\theta_m)$ and hence

$$|\tau_{M_N(\varepsilon)}(\theta_m) - \psi| \lesssim \frac{4^k}{M} + |I| \lesssim \frac{4^k}{M} \tag{3.11}$$

since $\tau_{M_N(\varepsilon)}(R_m)\cap I \neq \phi$. In view of Lemma 3

$$\text{mes}\,[\varepsilon; (3.11)] < \frac{1}{M} 4^k e^{C(\log N)^{1/2}}$$

by (2.34) and a suitable partition of the interval $[\psi - \frac{4^k}{M}, \psi + \frac{4^k}{M}]$. Hence, for k as above

$$\text{mes}\,[\varepsilon; \|M_N(\varepsilon)\| \sim 2^k \text{ and } \tau_{M_N(\varepsilon)}(R_m)\cap I \neq \phi] < e^{C(\log N)^{12}}\lambda. \tag{3.12}$$

Let $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By (2.2)

$$\tau'_{h^{-1}}(\theta) = \frac{1}{(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2}$$

and

$$\frac{1}{\|h\|^2} \lesssim \tau'_{h^{-1}}(\theta) \lesssim \min\Big(\frac{1}{\|h\|^2\|\theta-\theta_h\|^2}, \|h\|^2\Big) \tag{3.13}$$

for some $\theta_h \in \mathbb{T}$. Thus

$$\frac{1}{\|h\|^2} \lesssim D_m \lesssim \min\Big[\frac{1}{\|h\|^2\|\theta_m-\theta_h\|^2}, \|h\|^2\Big]. \tag{3.14}$$

Hence, given $D > 0$

$$\#\{1 \leq m \leq M; D_m \sim D\} \lesssim 1 + \frac{M}{\|h\| D^{1/2}}. \tag{3.15}$$

From (3.12), (3.15), we obtain the following estimate on (3.9)

$$(3.9) < e^{C(\log N)^{1/2}} \lambda(\log N) \Big\{ \max_{\|h\|^{-2} < D < \|h\|^2} \frac{M}{\|h\| D^{1/2}} \big(\max_{|J| < 4^k D|I|} \nu(J)\big)\Big\}$$

$$< e^{C'(\log N)^{1/2}} \big(\max_{\|h\|^{-2} < D < \|h\|^2} \frac{1}{D^{\frac{1}{2}} \|h\|} \max_{|J| < D|I|} \nu(J)\big) \tag{3.16}$$

since $k \lesssim (\log N)^{1/2}$ and writing J as a union of 4^k intervals of size at most $D.|I|$.

We distinguish several contributions

(i). For $D < \|h\|^{-1}$, estimate (3.16) by

$$e^{C'(\log N)^{\frac{1}{2}}} \max_{|J| < \frac{|I|}{\|h\|}} \nu(J) < e^{C'|\log \lambda|^{\frac{1}{2}}} \max_{|J| < \lambda^{\frac{1}{10}} |I|} \nu(J). \tag{3.17}$$

(ii) For $1 > D > \|h\|^{-1}$, we have $D^{1/2}\|h\| > \|h\|^{1/2} \gtrsim \lambda^{-\frac{1}{20}}$ and we may bound (3.16) by

$$\lambda^{\frac{1}{30}} \max_{|J| \leq |I|} \nu(J). \tag{3.18}$$

(iii) For $1 \leq D \leq \|h\|$, bound (3.16) by

$$e^{C'|\log \lambda|^{1/2}} \frac{D^{\frac{1}{2}}}{\|h\|} \max_{|J| \leq |I|} \nu(J) \leq \lambda^{\frac{1}{30}} \max_{|J| \leq |I|} \nu(J). \tag{3.19}$$

(iv) For $\|h\| < D < \|h\|^2$, estimate by

$$\frac{e^{C|\log \lambda|^{1/2}}}{D} \max_{|J| < D|I|} \nu(J). \tag{3.20}$$

Collecting the contributions (3.17) - (3.20) gives (3.5). This proves Lemma 4.

Next, returning to (3.3), writing $\mu = \frac{1}{2}\delta_{g_1} + \frac{1}{2}\delta_{g_2}$ we make the following construction. Assume

$$\nu = \int (\tau_g)_* [\nu] \mu_1(dg) \tag{3.21}$$

where μ_1 is some discrete probability measure $SL_2(\mathbb{R})$, such that

$$\|g\| < 2\lambda^{-\frac{1}{10}} \text{ for } g \in \operatorname{supp} \mu_1. \tag{3.22}$$

If $g \in \operatorname{supp} \mu_1$ and $\|g\| < \lambda^{-\frac{1}{10}}$, write by (3.3)

$$(\tau_g)_*[\nu] = \frac{1}{2}(\tau_{gg_1})_*[\nu] + \frac{1}{2}(\tau_{gg_2})_*[\nu].$$

Define then

$$\mu_2 = \sum_{\|g\| \geq \lambda^{-\frac{1}{10}}} \mu_1(g)\delta_g + \frac{1}{2} \sum_{\|g\| < \lambda^{-\frac{1}{10}}} \mu_1(g)(\delta_{gg_1} + \delta gg_2)$$

still satisfying (3.21).

From the positivity of the Lyapounov exponent, an iteration of this process will clearly produce a discrete probability measure $\tilde{\mu}$ on $SL_2(\mathbb{R})$ s.t.

$$\nu = \int (\tau_g)_*[\nu]\tilde{\mu}(dg) \tag{3.23}$$

and

$$\lambda^{-\frac{1}{10}} < \|g\| < 2\lambda^{-\frac{1}{10}} \text{ for } g \in \operatorname{supp} \tilde{\mu}. \tag{3.24}$$

Taking $N \sim \lambda^{-2}$, and since also by (3.2)

$$\nu = \int (\tau_{M_N(\varepsilon)})_*[\nu]d\varepsilon$$

(3.23) implies

$$\nu = \int \Big[\int (\tau_{M_N(\varepsilon)h})_*[\nu]d\varepsilon \Big] \tilde{\mu}(dh). \tag{3.25}$$

From(3.25) and Lemma 4, we conclude the following inequality.

Lemma 5. *For $I \subset \mathbb{T}$ an interval of size at most λ, we have*

$$\nu(I) \leq e^{C|\log \lambda|^{\frac{1}{2}}} \Big[\max_{|J| < \lambda^{\frac{1}{10}}|I|} \nu(J) + \max_{\lambda^{-\frac{1}{10}} < D < \lambda^{-\frac{1}{5}}} \frac{1}{D} \max_{|J| < D|I|} \nu(J) \Big]. \tag{3.26}$$

If we iterate (3.26) r-times, assuming $\lambda^{-\frac{r}{5}}|I| < \lambda$, we obtain

$$\nu(I) \leq 2^r e^{C|\log \lambda|^{\frac{1}{2}} r} \frac{1}{D_1} \nu(J) \tag{3.27}$$

for some interval J of size $|J| < D_1\delta_1|I|$ where $D_1 > 1, 0 < \delta_1 < 1$ and $D_1\delta_1^{-1} < \lambda^{-\frac{r}{10}}$.

From random matrix product theory it is known that the Furstenberg measure ν has some positive dimension $\alpha > 0$. Hence the right side of (3.27) is at most

$$\lesssim C^{|\log\lambda|^{\frac{1}{2}}r}\frac{1}{D_1}|J|^\alpha < C^{|\log\lambda|^{\frac{1}{2}}r}\frac{\delta_1^\alpha}{D_1^{1-\alpha}}|I|^\alpha < C^{|\log\lambda|^{\frac{1}{2}}r}\lambda^{\frac{r}{20}\ \min(\alpha,1-\alpha)}|I|^\alpha. \tag{3.28}$$

Assuming $\gamma > 0$ some constant (independent of $\lambda = o(1)$) satisfying

$$\gamma < \alpha < 1-\gamma \tag{3.29}$$

(3.28) and the restriction on r would imply for $\lambda < \lambda(\gamma)$

$$\nu(I) < (C^{|\log\lambda|^{\frac{1}{2}}}\lambda^{\frac{\gamma}{20}})^r|I|^\alpha < \lambda^{\frac{\gamma r}{30}}|I|^\alpha$$

and

$$\nu(I) < \Big(\frac{|I|}{\lambda}\Big)^{\frac{\gamma}{6}}|I|^\alpha \lesssim |I|^{\alpha+\frac{\gamma}{6}}. \tag{3.30}$$

But (3.30) would give that ν has dimension at least $\alpha + \frac{\gamma}{6}$, a contradiction.

Thus in order to prove

Theorem 1. *Assuming $\delta_0 < |E| < 2-\delta_0$, the dimension of the Furstenberg measure $\nu_E^{(\lambda)}$ for the A-B model is at least $\alpha(\lambda) \overset{\lambda\to 0}{\to} 1$.*

It will suffice to have a uniform lower bound in λ for $\dim \nu_E^{(\lambda)}$.

This is what we establish next.

Lemma 6. *Under the assumption of Theorem 1, $\dim \nu_E^{(\lambda)} > \gamma > 0$ with γ independent of λ.*

Proof. We make the following observation. Write $M = M_N(E;\varepsilon)$ as

$$M = \frac{(v_-^\perp \otimes v_+)\lambda_+ + (v_+^\perp \otimes v_-)\lambda_+^{-1}}{\langle v_+, v_-^\perp\rangle} \tag{3.31}$$

with v_+ (resp. v_-) the expanding (resp. contracting) direction. Hence

$$\|M\| \sim \frac{|\lambda_+|}{|v_+ \wedge v_-|}. \tag{3.32}$$

For unit vectors $u, w \in \mathbb{R}^2$, we deduce from (3.31) that

$$\frac{\|Mu\|}{\|M\|} = \|\langle v_-^\perp, u\rangle v_+ + \lambda_+^{-2}\langle v_+^\perp, u\rangle v_-\| =$$

$$(1+\lambda_+^{-2})|\langle v_-^\perp, u\rangle| + 0\Big(\frac{|v_+ \wedge v_-|}{\lambda_+^2}\Big) \overset{(3.32)}{\leq} (1+\lambda_+^{-2})|\langle v_-^\perp, u\rangle| + 0\Big(\frac{1}{\|M\|}\Big) \tag{3.33}$$

and

$$\begin{aligned}\frac{|\langle Mu, w\rangle|}{\|M\|} &= |\langle v_-^\perp, u\rangle\langle v_+, w\rangle + \lambda^{-2}\langle v_+^\perp, u\rangle\langle v_-, w\rangle| \\ &\geq (1+\lambda^{-2})|\langle v_-^\perp, u\rangle|\ |\langle v_+, w\rangle| - 2\lambda_+^{-2}|v_+ \wedge v_-| \\ &\geq |\langle v_-^\perp, u\rangle|\ |\langle v_+, w\rangle| + 0\Big(\frac{1}{\|M\|}\Big).\end{aligned} \tag{3.34}$$

Hence, given an arc I of size η centered at ν

$$\begin{aligned}&\mathbb{P}\Big[\varepsilon; v_- \in I \text{ where } v_- \text{ is contracting direction of } M_N(\varepsilon)\Big] \leq^{(3.33)} \\ &\mathbb{P}\Big[\varepsilon; \frac{\|M_N(\varepsilon)u\|}{\|M_N(\varepsilon)\|} < 2\eta + 0\Big(\frac{1}{\|M_N(\varepsilon)\|}\Big)\Big] \leq \\ &\mathbb{P}\big[\varepsilon; \|M_N(\varepsilon)\| < e^{\frac{\lambda^2}{20}N}\Big] + \mathbb{P}\Big[\varepsilon; \frac{\|M_N(\varepsilon)u\|}{\|M_N(\varepsilon)\|} < 3\eta\Big] = \\ &(3.35) + (3.36)\end{aligned}$$

provided

$$\eta > e^{-\frac{\lambda^2}{20}N}. \tag{3.37}$$

Recalling (2.29), (2.30), we have

$$\operatorname{mes}\Big[\varepsilon\ \Big|\frac{1}{N}\frac{\log\|M_N(\varepsilon)\|}{L(E)} - 1\Big| > a\Big] < e^{\big(-\frac{a^2}{2}L(E) + 0(\lambda^3)\big)N} \tag{3.38}$$

with $\frac{\lambda^2}{8} < L(E) < 0(\lambda^2)$.

Hence,

$$(3.35) < e^{-(\frac{1}{50}\lambda^2 + 0(\lambda^3))N} < e^{-\frac{1}{60}\lambda^2 N} \overset{(3.37)}{<} \eta^{\frac{1}{3}} \tag{3.39}$$

for λ small enough.

Next, we point out that in the analysis (2.14)-(2.28), the formula (2.28) is equally valid for $\frac{1}{N}\log\|M_N(E;\varepsilon)(u)\|$, with $u\in S^1$ arbitrary (as a consequence of the argument). Thus we can write

$$\frac{1}{N}\log\|M_N(\varepsilon)\| = L(E) - \frac{\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n d_n + 0(\lambda^3) \tag{3.40}$$

and

$$\frac{1}{N}\log\|M_N(\varepsilon)(u)\| = L(E) - \frac{\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n d_n' + 0(\lambda^3) \tag{3.41}$$

so that

$$\log\frac{\|M_N(\varepsilon)\|}{\|M_N(\varepsilon)(u)\|} = \frac{\lambda}{2\sin\kappa}\sum_1^N \varepsilon_n(d_n' - d_n) + 0(N\lambda^3) \tag{3.42}$$

where d_n, d_n' depend on $\varepsilon_1,\dots,\varepsilon_{n-1}$.

Letting $1>t>0$ be a parameter, write

$$(3.36) < (3\eta)^t\int\Big(\frac{\|M_N(\varepsilon)\|}{\|M_N(\varepsilon)(u)\|}\Big)^t d\varepsilon < (3\eta)^t e^{0(N\lambda^3 t)}\int e^{\frac{\lambda t}{2\sin\kappa}\sum_1^N\varepsilon_n(d_n'-d_n)}d\varepsilon$$

$$< (3\eta)^t e^{0(N\lambda^3 t)}e^{C\lambda^2 t^2 N} \tag{3.43}$$

where the constant C only depends on E.

Choosing N s.t.

$$\eta\sim e^{-\frac{\lambda^2}{10^3}N} \tag{3.44}$$

we satisfy (3.37), and it follows from (3.43) and appropriate choice of t, that

$$(3.36) < (3\eta)^{t-C(\lambda t+t^2)} < \eta^{c_1}$$

(again for λ small enough) and with $c_1>0$ independent of λ.

Hence, we showed that with N satisfying (3.44)

$$\text{mes}\,[\varepsilon; v_-\in I \text{ where } v_- \text{ is contracting vector of } M_N(\varepsilon)] < \eta^{c_1}. \tag{3.45}$$

Since v_+ is the contracting vector of $M_N(\varepsilon)^{-1}$, we obtain a similar statement for the expanding vector. Therefore, given any pair of η-intervals I_+, I_- in S^1, we proved that

$$\begin{aligned}&\text{mes}\,[\varepsilon; v_+\in I_+, v_-\in I_-\\ &\quad\text{with } v_+ \text{ (resp. } v_-\text{) expanding (resp. contracting) direction of } M_N(\varepsilon)] < 2\eta^{c_1}\end{aligned} \tag{3.46}$$

for N satisfying (3.44).

Returning to (3.34), we have

$$\begin{aligned}&\mathbb{P}[\varepsilon;\frac{|\langle M_N(\varepsilon)u,w\rangle|}{\|M_N(\varepsilon)\|}<\eta_1]\leq\\&\mathbb{P}[\varepsilon;\|M_N(\varepsilon)\|<1/\eta_1]+\mathbb{P}[\varepsilon;|\langle v_-^\perp,u\rangle|\lesssim\sqrt{\eta_1}]+\mathbb{P}[\varepsilon;|\langle v_+,w\rangle|\lesssim\sqrt{\eta_1}].\end{aligned}\tag{3.47}$$

Taking $\eta=\eta_1^{\frac{1}{2}}$ and N as in (3.44), the last 2 terms in (3.37) are at most $0(\eta_1^{\frac{1}{2}c_1})$ by (3.46), while the first term is bounded by mes $[\varepsilon;\|M_N(\varepsilon)\|<e^{\frac{1}{500\lambda^2 N}}]<e^{-\frac{1}{60}\lambda^2N}<\eta_1$ by (3.38).

Hence

$$(3.47)\lesssim\eta_1^{\frac{1}{2}c_1}\text{ with }\eta_1\sim e^{-\frac{\lambda^2}{500}N}.\tag{3.48}$$

Returning to the Furstenberg measure $\nu=\nu_E^{(\lambda)}$, we have for $I\subset\mathbb{T}$ a small arc of size η_1, by (3.4)

$$\nu(I)=\lim_{N'\to\infty}\mathbb{P}\Big[\varepsilon|\frac{M_{N'}(\varepsilon)e_1}{\|M_{N'}(\varepsilon)e_1\|}\in I\Big].$$

Take N as in (3.48) and $N'>N$. If w denotes the center of I, then

$$\frac{|\langle M_{N'}e_1,w^\perp\rangle|}{\|M_{N'}e_1\|}<\eta_1.\tag{3.49}$$

Fix $\varepsilon_1,\dots,\varepsilon_{N'-N}$ and let $u=\frac{M_{N'-N}(\varepsilon_1,\dots,\varepsilon_{N'-N})(e_1)}{\|M_{N'-N_1},\dots,(\varepsilon_{N'-N})(e_1)\|}$. We have

$$\frac{M_{N'}e_1}{\|M_{N'}e_1\|}=\frac{M_N(\varepsilon_{N'-N+1},\dots,\varepsilon_{N'})(u)}{\|M_N(\varepsilon_{N'-N+1},\dots,\varepsilon_{N'})u\|}.$$

Thus (3.49) implies

$$\frac{|\langle M_N(\cdots)u,w^\perp\rangle|}{\|M_N(\cdots)\|}<\eta_1\tag{3.50}$$

for which the measure in $\varepsilon_{_{N'-N+1}},\dots,\varepsilon_{N'}$ is at most $\eta_1^{\frac{1}{2}c_1}$ by(3.48). Therefore

$$\nu(I)\lesssim|I|^{\frac{1}{2}c_1}.\tag{3.51}$$

This proves that $\dim\nu\geq\frac{1}{2}c_1$, uniformly in λ. Hence we establish Lemma 6.

This also completes the proof of Theorem 1.

§4. Density of states

Let $u, w \in S^1$, $\eta > 0$ small. It follows from (3.34) that

$$\lim_{N\to\infty} \text{mes}\left[ve; \frac{|\langle M_N(\varepsilon)u, w\rangle|}{\|M_N(\varepsilon)\|} < \eta\right] \leq$$

$$\lim_{N\to\infty} \text{mes}\left[\varepsilon; |\langle v_+, w\rangle|.|\langle v_-^\perp, u\rangle| < \eta \text{ with } v_+, v_- \text{ the eigenvectors of } M_N(\varepsilon)\right] =$$

$$\lim_{N\to\infty} \text{mes}\Big[(\varepsilon, \varepsilon'); |\langle v_+, w\rangle|.|\langle v'_+, u^\perp\rangle| < \eta \text{ with } v_+ \text{ (resp } v'_+\text{) expanding direction of}$$

$$M_N(\varepsilon), \text{ (resp } M_N(\varepsilon'))\Big]$$

$$\lesssim \log\frac{1}{\eta}. \max_{\eta_1\cdot\eta_2=\eta} \nu_E(I_{\eta_1}(w^\perp)).\nu_E(I_{\eta_2}(u)) \ll \eta^\gamma. \tag{4.1}$$

where we used (3.4) and the independence of v_+, v_- for $N \to \infty$ as functions of ε.

Here $\gamma < \dim \nu_E^{(\lambda)}$ and $\gamma = \gamma(\lambda) \to 1$ for $\lambda \to 0$.

It is easily seen that (4.1) implies that for given $K > 1$ and taking N large enough (depending on K)

$$\max_{u,w\in S^1} \mathbb{E}\Big[\frac{\|M_N\|}{|\langle M_N u, w\rangle|} \wedge K\Big] < K^{1-\gamma}. \tag{4.2}$$

Here $M_N = M_N(E)$ and (4.2) remains clearly valid replacing E by $z = E + iy$ with $0 < y < y_N$ small enough (depending on N) and taking for u, w unit vectors in $\mathbb{C}^2$.

Next, take $N' > N$ and consider

$$\frac{\|M_{[0,N']}(z;\varepsilon)\|.\|M_{]1N',2N']}(z;\varepsilon)\|}{\|M_{[0,2N']}(z,\varepsilon)\|}. \tag{4.3}$$

Fixing $\varepsilon_{N'+1}, \dots, \varepsilon_{2N'}$, we obtain a unit vector $\zeta \in \mathbb{C}^2$ (depending on these variables) such that

$$(4.3) = \frac{\|M_{[0,N']}(z,\varepsilon)\|}{\|M_{[0,N']}(z,\varepsilon)(\zeta)\|} \tag{4.4}$$

and

$$(4.4) \lesssim \sum_{i,j=1,2} \frac{|\langle M_{[0,N']}(z,\varepsilon)e_i, e_j\rangle|}{|\langle M_{[0,N'']}(z,\varepsilon)\zeta, e_j\rangle|}.$$

Fix also $\varepsilon_1, \dots, \varepsilon_{N'-N}$ and let ζ_1 be a unit vector in $\mathbb{C}^2$ with ζ_1 parallel to $M^*_{[0,N'-N]}(z,\varepsilon)e_j$. Hence

$$\frac{|\langle M_{[0,N']}(z,\varepsilon)e_i, e_j\rangle|}{|\langle M_{[0,N']}(z,\varepsilon)\zeta, e_j\rangle|} \leq \frac{\|M_{[N'-N+1,N']}(z,\varepsilon)\|}{|\langle M_{[N'-N+1,N']}(z,\varepsilon)\zeta, \zeta_1\rangle|} \tag{4.5}$$

where the vectors ζ, ζ_1 do not depend on $\varepsilon_{N'-N+1}, \dots, \varepsilon_{N'}$.

Thus

$$\min\big((4.3), K\big) \lesssim \min\big((4.5), K\big). \tag{4.6}$$

Taking expectation of (4.6) in $\varepsilon_{N'-N+1}, \dots, \varepsilon_{N'}$ (with other variables fixed), (4.2) and subsequent remark, give an estimate

$$\mathbb{E}_{\varepsilon_{N'-N+1}, \dots, \varepsilon_{N'}}[(4.6)] \lesssim K^{1-\gamma}. \tag{4.7}$$

Hence, also

$$\mathbb{E}[\min\big((4.3), K\big)] \lesssim K^{1-\gamma} \tag{4.8}$$

valid for $z = E + iy$ with $y > 0$ small enough (depending on K) and $N' > N'(K)$.

Denoting $\mathcal{N}$ the IDS, recall that

$$\bar{\partial}\mathcal{N}(z) = \mathbb{E}[G(0,0,z)] \qquad z = E + iy$$

where $G(z) = (H - z)^{-1}$ is the Green's function and $\mathcal{N}(z)$ the harmonic extension of $\mathcal{N}$ to $\operatorname{Im} z > 0$.

Fix $z, \operatorname{Im} z > 0$. Then from the resolvent identity and positivity of the Lyapounov exponent, we obtain

$$G(0,0,z) = \lim_{\substack{\Lambda = [-a,b] \\ a,b \to \infty}} G_\Lambda(0,0,z) \quad \text{a.s}$$

and, by Cramer's rule

$$|G(0,0,z)| \le \underline{\lim}_{N' \to \infty} \frac{\|M_{[-N',0]}(z,\varepsilon)\| \ \|M_{[0,N']}(z,\varepsilon)\|}{\|M_{[-N',N']}(z;\varepsilon)\|}. \tag{4.9}$$

Hence by (4.8)

$$\mathbb{E}[|G(0,0,z)| \wedge K] \le \underline{\lim}_{N' \to \infty} \mathbb{E}[\cdots \wedge K] \lesssim K^{1-\gamma} \tag{4.10}$$

if $y > 0$ is small enough (depending on K). Letting $y \to 0$ we get

$$\mathbb{E}[|G(0,0,E+io)| \wedge K] < K^{1-\gamma}. \tag{4.11}$$

It follows from (4.11) that for $0 < \gamma_1 < \gamma$

$$\mathbb{E}[|G(0,0,E+io)|^{\gamma_1}]R < C. \tag{4.12}$$

Recall that we assumed $\delta_0 < |E| < 2 - \delta_0$. Using the subharmonicity of $|G(0,0,z)|^{\gamma_1}$ on $\operatorname{Im} z > 0$, we deduce from (4.12) that for fixed $z = E + iy, y > 0$

$$|\bar{\partial}\mathcal{N}(z)| \le \mathbb{E}[|G(0,0,z)|] < \frac{1}{y^{1-\gamma_1}} \mathbb{E}[|G(0,0,z)|^{\gamma_1}] < \frac{C}{y^{1-\gamma_1}}. \tag{4.13}$$

Hence $\mathcal{N}$ is γ_1-Hölder for all $\gamma_1 < \gamma$.

This proves

Theorem 2. *For $\delta_0 < |E| < 2 - \delta_0$, the IDS of the A-B model with λ-disorder is s-Hölder regular, with $s \to 1$ for $\lambda \to 0$.*

§5. Further comments

If one aims at going further and prove the Lipschitz regularity of the IDS, it seems reasonable to prove that the Furstenberg measures on the projective line $P_1(\mathbb{R}) \simeq \mathbb{T}$ are at least absolutely continuous. This is far from an obvious issue. In fact, it was conjectured in [K-L] that if ν is a finitely supported probability measure on $SL_2(\mathbb{R})$, then its Furstenberg measure on $P_1(\mathbb{R})$ is always singular. This conjecture was disproved in [BPS] using a probabilistic construction reminiscent of random Bernoulli-convolutions. An explicit example was given recently in [B2], based on a construction from [B3] $\big($that relies on an extension of the spectral gap theory from [BG1] for $SU(2)$ to $SL_2(\mathbb{R})\big)$. A rough description is as follows. One produces a finite subset $\mathcal{G} \subset SL_2(\mathbb{R}) \cap \mathrm{Mat}_{2\times 2}(q)$, q a fixed large integer, such that $\log(\#\mathcal{G}) \sim \log q$, $\mathcal{G}$ generates freely the free group on $\#\mathcal{G}$ generators and moreover $\mathcal{G}$ is contained in a small neighborhood of the identity (depending on q). Denoting

$$\nu = \frac{1}{(\#\mathcal{G})} \sum_{g\in\mathcal{G}} \delta_g \tag{5.1}$$

the probability measure on $SL_2(\mathbb{R})$, it is shown that there is a spectral gap for the projective representation ρ, in the following sense. Let $f \in L^2(\mathbb{T})$, $\|f\|_2 = 1$ and assume $\hat{f}(n) = 0$ for $|n| > K$, where $K = K(q)$ is a sufficiently large constant. Then

$$\frac{1}{(\#\mathcal{G})} \Big\| \sum_{g\in\mathcal{G}} \rho_g f \Big\|_2 < \frac{1}{2} \tag{5.2}$$

where $\rho_g f = (\tau_g')^{-\frac{1}{2}}(f \circ \tau_g)$ and τ_g the action on $\mathbb{T}$ defined for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by

$$e^{i\tau_g(\theta)} = \frac{(a\cos\theta + b\sin\theta) + i(c\cos\theta + d\sin\theta)}{[(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2]^{\frac{1}{2}}}. \tag{5.3}$$

Since $g \in \mathcal{G}$ are close to identity, (5.2) clearly implies that for f as above

$$\frac{1}{(\#\mathcal{G})} \Big\| \sum_{g\in\mathcal{G}} (f \circ \tau_g) \Big\|_2 < \frac{3}{4}. \tag{5.4}$$

From (5.4), one may then derive easily that ν has an a.c. Furstenberg measure with C^k-density, where k can be made arbitrarily large.

It should be pointed out that the contractive properties (5.2), (5.4) are not exploiting hyperbolicity (at least in the usual sense), as the Lyapounov exponent of the random matrix product corresponding to ν is small.

Returning to the A-B-model with small λ, denote

$$\nu_{\lambda,E} = \frac{1}{2}\delta_{\begin{pmatrix} E+\lambda & -1 \\ 1 & 0 \end{pmatrix}} + \frac{1}{2}\delta_{\begin{pmatrix} E-\lambda & -1 \\ 1 & 0 \end{pmatrix}} \tag{5.5}$$

and $\nu^{(\ell)}_{\lambda,E}$ its ℓ-fold convolution. It seems reasonable to believe that

$$\Big\| \sum_g \nu^{(\ell)}_{\lambda,E}(g)(f\circ\tau_g)\Big\|_2 \leq \frac{1}{2}\|f\|_2 \tag{5.6}$$

for $f\in L^2(\mathbb{T}), \hat{f}(n)=0$ for $|n|>K(\lambda)$ and where ℓ is some positive integer independent of λ, or at least $\ell = o(\lambda^{-2})$. Such property would then again imply a.c. and a certain smoothness of the Furstenberg measure. Unfortunately, available technology to establish spectral gaps (as developed in [BG1]) so far requires algebraic matrix elements of bounded height and hence does not apply to (5.5).

One may however combine the methods from [BG1] with those of [S-T] to prove the following result, which seems new (compare also with the results from [K-S]).

Theorem 3. *Consider a random Schrödinger operator $H=\Delta+V$ on $\mathbb{Z}$ where $V=(V_n)_{n\in\mathbb{Z}}$ are i.i.d's with distribution given by a compactly supported measure β on $\mathbb{R}$ of positive dimension. Thus there is $\kappa>0$ s.t.*

$$\beta(I)\lesssim |I|^\kappa \text{ for } I\subset\mathbb{R} \text{ an interval.} \tag{5.7}$$

Then H has C^∞ density of states.

We sketch the argument.

For fixed E, let μ_E be the probability measure on $SL_2(\mathbb{R})$ obtained as image measure of β under the map

$$v\mapsto \begin{pmatrix} E-v & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.8}$$

Following [S-T], it will suffice to show that, for some fixed convolution power ℓ, the measure $\mu_1=\mu_E^{(\ell)}$ on $SL_2(\mathbb{R})$ gives a smoothing convolution operator on $P_1(\mathbb{R})$. Thus there is some $\alpha>0$ s.t. for $f\in H^s(\mathbb{T}), s\geq 0$

$$\Big\|\int (f\circ\tau_g)\mu_1(dg)\Big\|_{H^{s+\alpha}} \lesssim \|f\|_{H^s} \tag{5.9}$$

(where H' denotes the usual Sobolev space with norm $\|f\|_{H^s} = \big(\sum(1+|n|)^{2s}|\hat{f}(n)|^2\big)^{\frac{1}{2}}\big)$.

Denoting $x = E - v$, one has

$$\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} xyz - x - z & 1 - xy \\ yz - 1 & -y \end{pmatrix} \tag{5.10}$$

and recalling (5.7), one sees that $\mu_E^{(3)}$ certainly has the property that

$$\mu_E^{(3)}(\mathfrak{S}_\delta) \lesssim \delta^{\kappa'} \text{ for all } \delta > 0 \tag{5.11}$$

if $\mathfrak{S}$ is a proper algebraic subvariety of $SL_2(\mathbb{R})$ of bounded degree and $\mathfrak{S}_\delta$ denotes a δ-neighborhood of $\mathcal{G}$. Here $\kappa' > 0$ depends on the degree bound.

Let P_δ, $\delta > 0$, denote an approximate identity on $SL_2(\mathbb{R})$. Using (5.11), an extension of the 'flattening Lemma' from [BG1] to $SL_2(\mathbb{R})$ (note that, up to complexification, $SU(2)$ and $SL_2(\mathbb{R})$ have the same Lie-algebra and our analysis is local), permits us to conclude the following.

Lemma 7. *Fix some $0 < \varepsilon < 1$. There is $\ell = \ell(\varepsilon) \in \mathbb{Z}_+$ s.t. for all $\delta > 0$, we have*

$$\|\mu_E^{(3\ell)} * P_\delta\|_\infty < \delta^{-\varepsilon} \tag{5.12}$$

(in particular, $\mu_E^{(3\ell)}$ has dimension at least $3 - \varepsilon$).

This is the crucial step, depending on 'arithmetic combinatorics' in groups (see [BG1] and related refs for more details).

Taking $\varepsilon = 10^{-3}$ and $\ell = \ell(\varepsilon)$ given by Lemma 7, we can now prove that $\mu_1 = \mu_E^{(3\ell)}$ satisfies (5.9). This will clearly be a consequence of the following statement.

Lemma 8. *Let $f \in L^2(\mathbb{T}), \|f\|_2 = 1$ and supp $\hat{f} \subset [2^k, 2^{k+1}] \cup [-2^{k+1}, -2^k]$ with k sufficiently large. Then*

$$\Big\| \int (f \circ \tau_g)\mu_1(dg)\Big\|_2 < 2^{-k\kappa}. \tag{5.13}$$

for some $\kappa > 0$.

Proof of Lemma 8.

We summarize the argument from [B2].

Denote $G = SL_2(\mathbb{R})$ and take $\delta = 4^{-k}$, so that, by assumption on f, we may replace the left side of (5.13) by

$$\Big\| \int (f \circ \tau_g)(\mu_1 * P_\delta)(dg) \Big\|_2. \tag{5.14}$$

Using (5.12), one gets

$$(5.14)^2 \lesssim \delta^{-2\varepsilon} \iint_{G\times G} |\langle f \circ \tau_{g_1}, f \circ \tau_{g_2}\rangle| \Omega(g_1)\Omega(g_2) dg_1 dg_2$$

with $0 \le \Omega \le 1$ a suitable compactly supported function on G (depending on the support of β). Next, by Cauchy-Schwarz

$$(5.14)^4 \lesssim \delta^{-4\varepsilon} \iiiint_{G\times G\times \mathbb{T}\times\mathbb{T}} f(\tau_{g_1}x)\bar{f}(\tau_{g_2}x)f(\tau_{g_1}y)f(\tau_{g_2}y)\Omega(g_1)\Omega(g_2)dg_1dg_2dxdy. \tag{5.15}$$

To estimate (5.15), proceed as follows. Fix $x, y \in \mathbb{T}$ and $g_1 \in G$ and consider the integral in g_2

$$\int \bar{f}(\tau_g x) f(\tau_g y)\Omega(g)dg. \tag{5.16}$$

The point here is that if one specifies $\tau_g x \in \mathbb{T}$, there remains an average in $\tau_g y$ to be exploited, when integrating in g (unless x and y are very close). More precisely, if $\|x-y\| < 2^{-h/10}$, then

$$|(5.16)| < 2^{-k}\|f\|_1^2$$

and the contribution in (5.15) is at most

$$\le \delta^{-4\varepsilon}2^{-k}\|f\|_1^4 < 2^{-k/2}.$$

The contribution of $\|x-y\| < 2^{-k/10}$ in (5.15) is easily estimated by

$$\iint_{\|x-y\|<2^{-k/10}} \Big[\int_G |f(\tau_g x)|\ |f(\tau_g y)|\Omega(g)dg\Big]^2 dxdy \le$$
$$\iint_{\|x-y\|<2^{-k/10}} \Big[\int_G |f(\tau_g x)|^2\Omega(g)dg\Big]\Big[\int_G |f(\tau_g y)|^2\Omega(g)dg\Big]dxdy$$

$$\lesssim 2^{-k/10}\|f\|_2^4$$

and (5.13) follows.

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